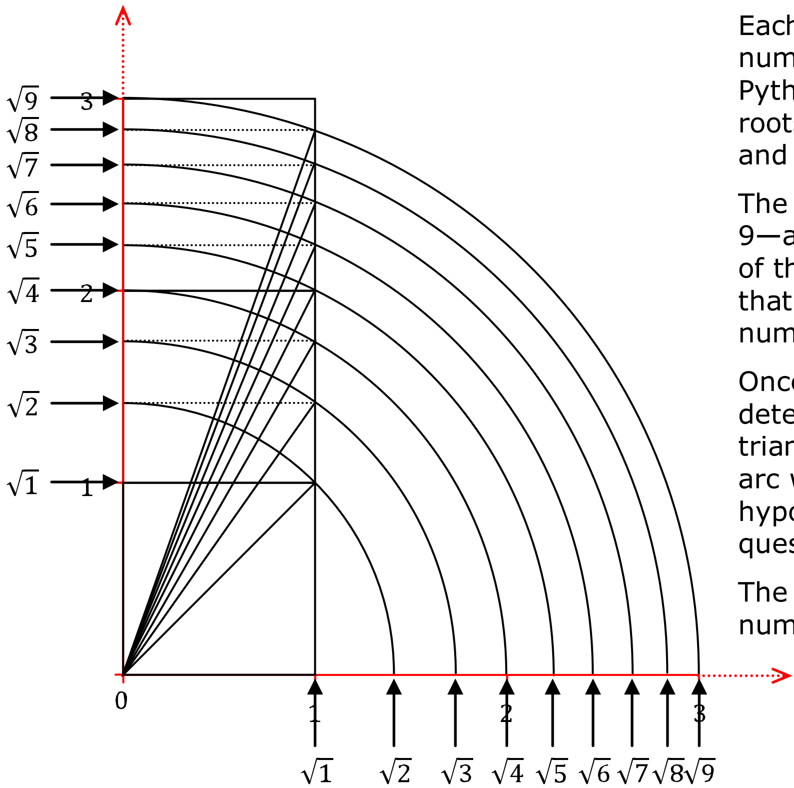
number line ruminations:

reconsidering the root spiral of Whozitz



Each of the roots has its unique home on the number line. But where? Thanks to the Pythagorean Theorem, we can locate square roots on the number line using a unit square and a compass.

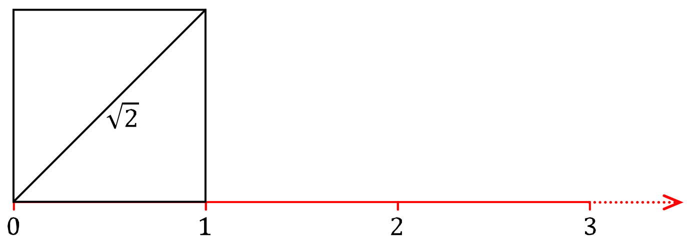
The graphic on the left shows roots 2 through 9—as hypotenuses of right triangles with legs of the preceding root and the unit length, 1, that defines both the unit square and the number line.

Once we have the length of the square root determined as the hypotenuse of a right triangle, we'll use the compass to swing an arc with the origin at its center and the hypotenuse of each right triangle—the root in question—as its radius.

The points where the arcs intersect the number line locate each root.

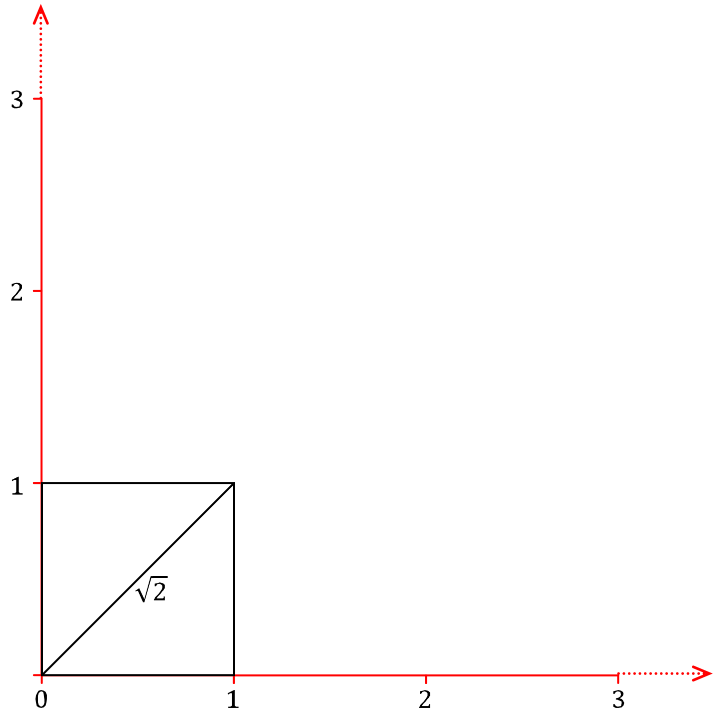
To build the diagram above, we'll start with our old friend, the number line—just a 3-unit chunk of it:

Now we'll plunk a unit square on it and draw a diagonal, starting at the origin. Since the legs of the right triangles formed by a diagonal of the square are each 1 unit, by the Pythagorean Theorem, the length of its diagonal will be $\sqrt{2}$.

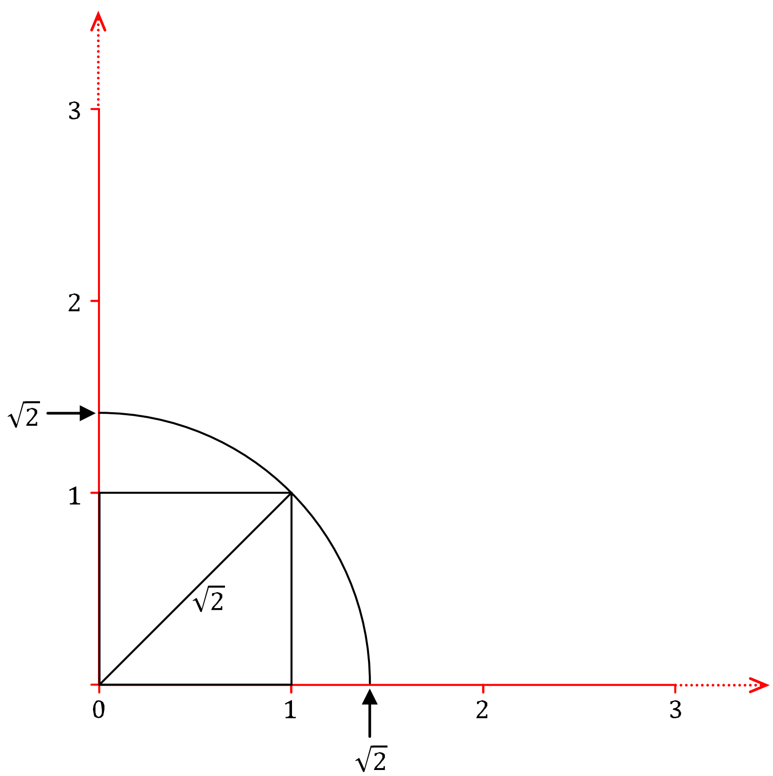


(That's one root down, infinity to go.)

Since we’ve got a unit square there anyway, starting at the origin, we can copy our chunk of the number line, and send it up vertically:



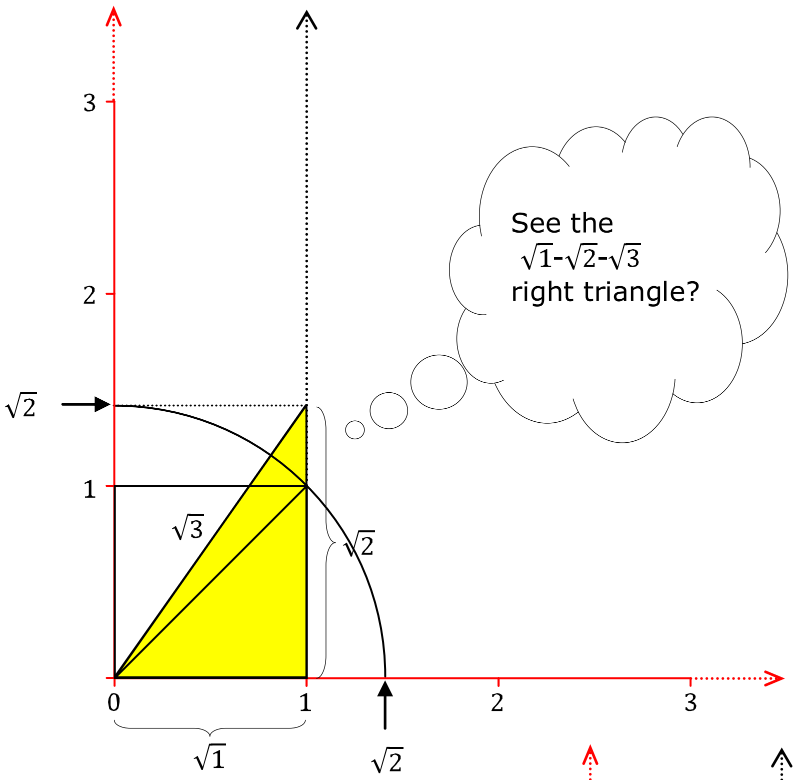
Where does the $\sqrt{2}$ lie on the number line? All we have to do is swing an arc whose radius is equal to the diagonal, $\sqrt{2}$, and whose center is at the origin:



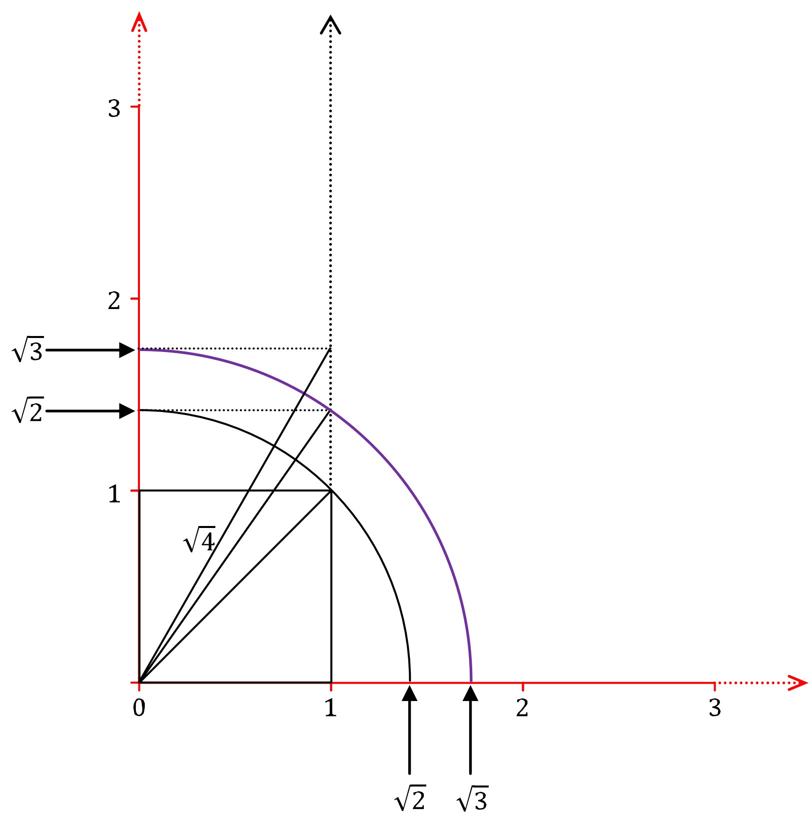
Now we’ve located $\sqrt{2}$ on the number line. We can say that the length $\sqrt{2}$ is the distance between the origin, zero, and the point we’re calling $\sqrt{2}$, the distance $[0, \sqrt{2}]$

This will be our method—the Pythagorean Theorem and circles with their radii equal to the hypotenuse. We'll be making new right triangles having the previous root as one leg and the other leg being our old buddy, 1 unit (also, conveniently, equalling $\sqrt{1}$). The hypotenuse will be equal to the new root which we then locate on the horizontal and vertical number lines with an arc.

Now for $\sqrt{3}$. If we go ahead and locate $\sqrt{2}$ on our vertical number line and extend a 1-unit segment perpendicular to the vertical number line at that point, we have the legs of a right triangle, the hypotenuse of which will be $\sqrt{3}$, by the Pythagorean Theorem: $1^2 + \sqrt{2}^2 = \sqrt{3}^2$ which simplifies to $1 + 2 = 3$

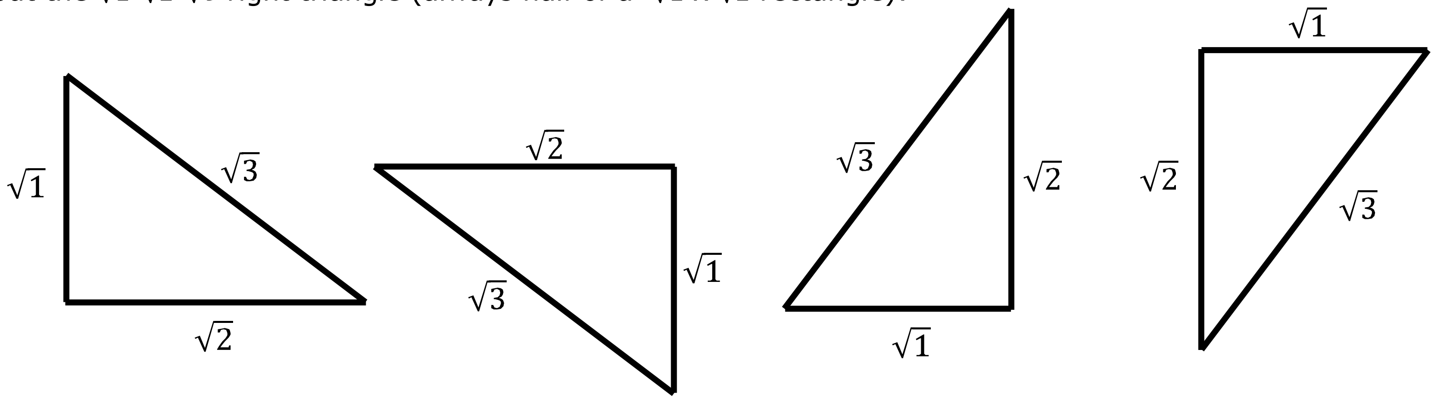


We can then swing another arc, locating $\sqrt{3}$ on our number lines, and proceed on up towards infinity, locating as each root on the number line as we go with concentric circles, then extend a one-unit leg for each new triangle and then forming the hypotenuse by connecting its end-point to the origin.

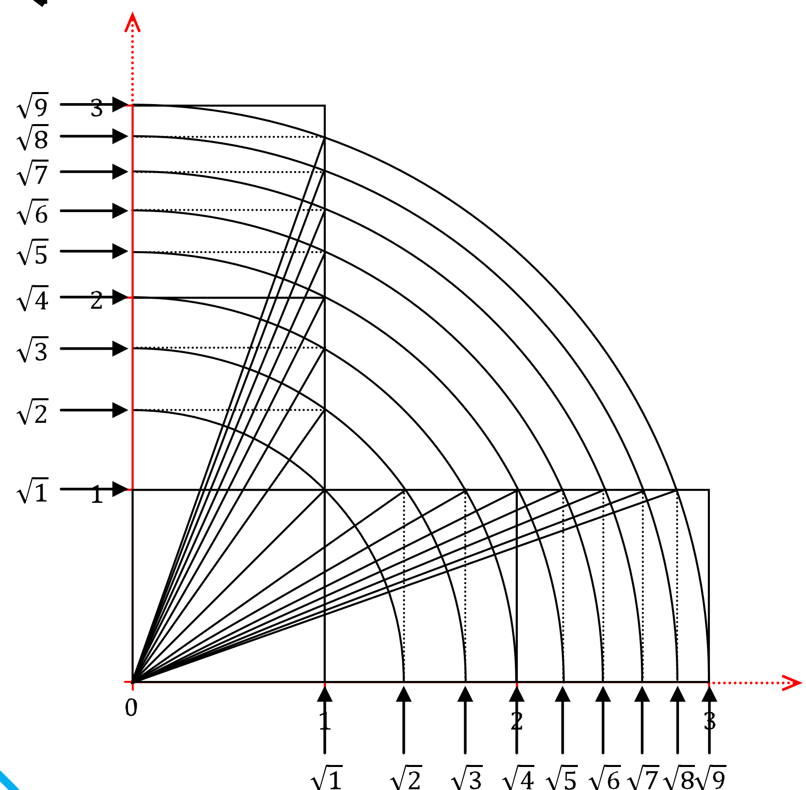


A handy icon for me in all this was not so much $a^2 + b^2 = c^2$,

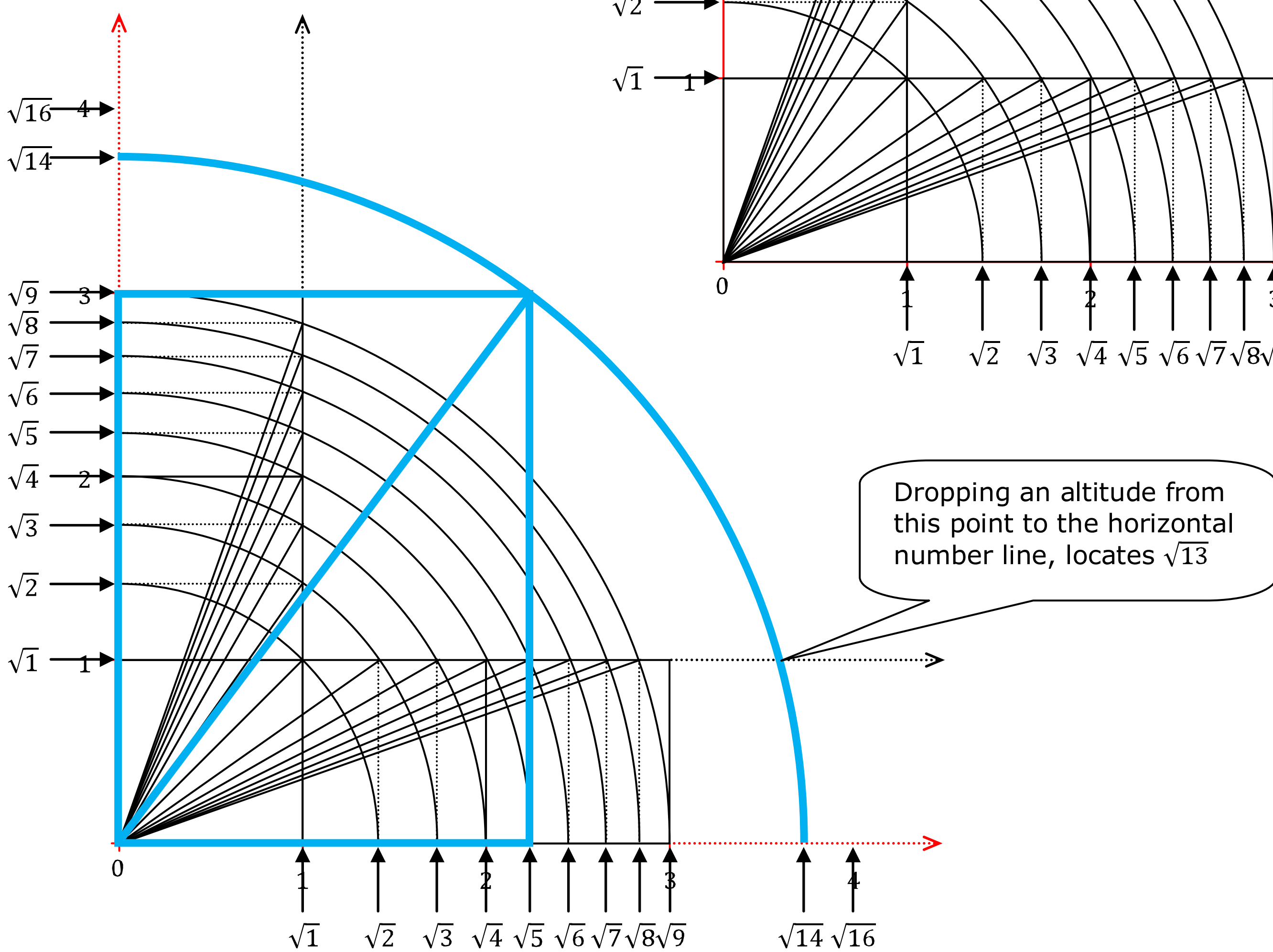
but the $\sqrt{1}$-$\sqrt{2}$-$\sqrt{3}$ right triangle (always half of a $\sqrt{1}$ x $\sqrt{2}$ rectangle):



We don't have to progress vertically;
we could just as easily go horizontally:

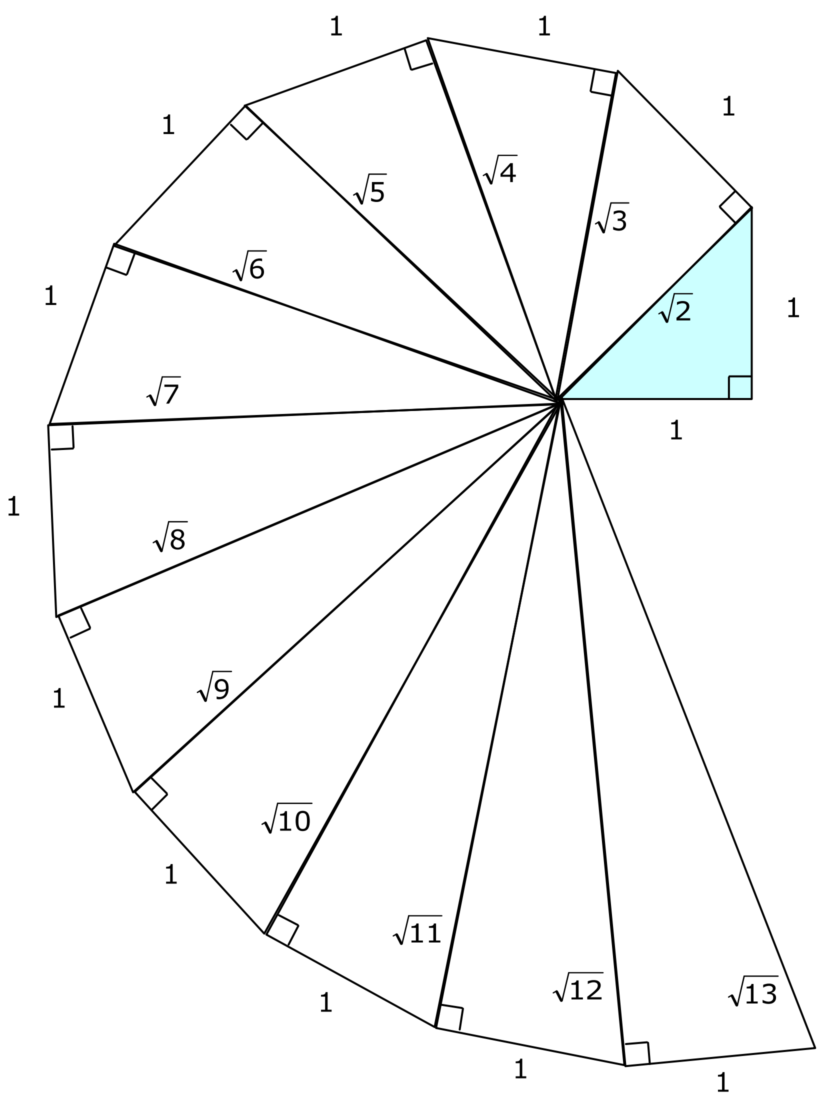


And we don't have to just make these roots one-by-one. If we want to locate $\sqrt{14}$ on the number line, for example, we can use whatever size rectangle we like—just so the legs add up. Below we make a $\sqrt{9}$ x $\sqrt{5}$ rectangle and the diagonal of the rectangle will be $\sqrt{14}$



There is, of course, the root spiral of Theodorus—very pretty, to be sure, but no number line. And here are some pertinent links:

- "Investigations: Compass and Straightedge Constructions" gives a nice overview/illustrations of constructions in Euclidean Geometry and also links you to interactive Java Sketchpad and Geometer's Sketchpad explorations—including something a lot like the foregoing at http://jwilson.coe.uga.edu/EMAT6680Fa06/Love/Investigation.htm
- "Building Square Roots and Multiples from Unit Squares" leads you through a construction of a few special triangles and finally the the Root Spiral of Theodorus at www.soesd.k12.or.us/files/building_roots_3.pdf, and
- You can find a nice java animation of the root spiral at www.csua.berkeley.edu/~raytrace/java/spirals/square.html



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