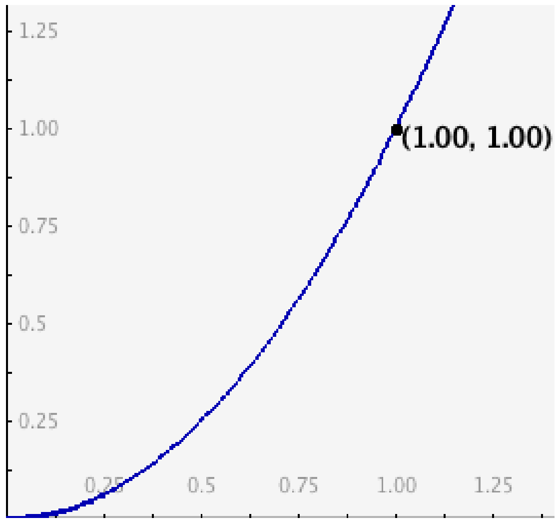
Area under a parabola

Suppose we're given the chore of finding the area under a parabola, $y = x^2$, between 0 and 1:



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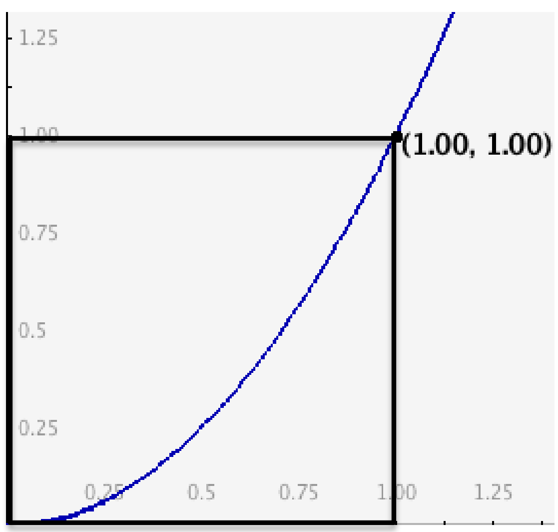
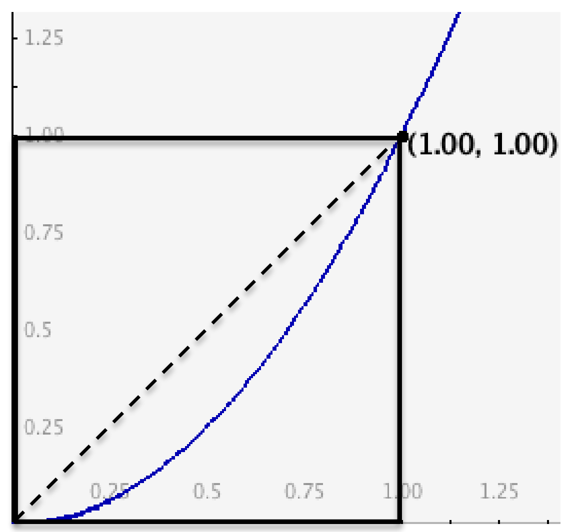
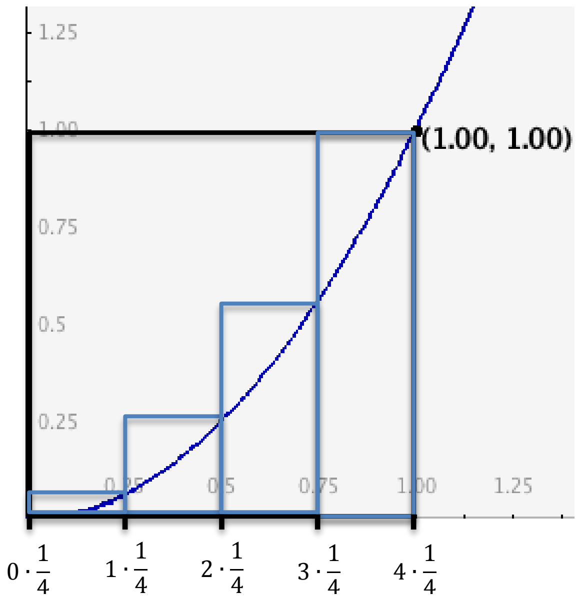
We can tell the area under the parabola will be less than 1 square unit and even less than 1/2 square unit by imagining or superimposing a unit square and even drawing a diagonal:

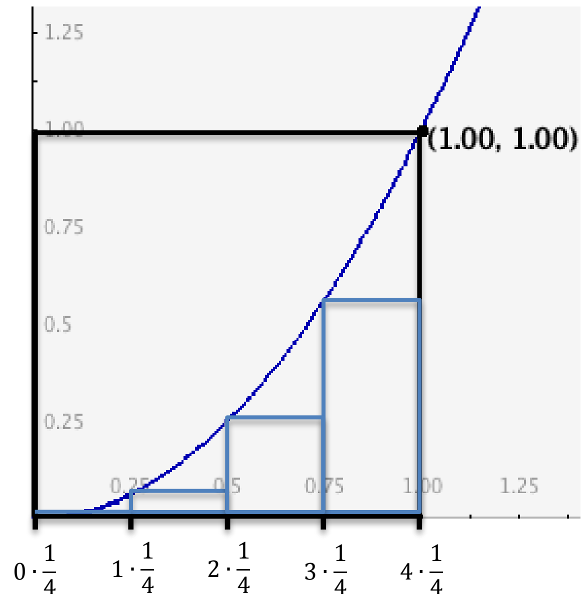




We can say with confidence that the area under the parabola between 0 and 1 is less than 1/2 square unit. That's something, but we can do better.

We know how to find the area of rectangles, so let's try making some rectangles and use them to start to get a handle on this problem.





We could divide the segment [0, 1] into 4 equal segments and consider the approximate area under the curve to be roughly equal to the sum of the areas of the 4 rectangles we create with base equal to 1/4 of segment [0, 1] and height equal to the distance between the x-axis and the parabola through the endpoint of the base. The bases of the rectangles are all the same, but what do we call the height? We can use our function $y = x^2$ to calculate the height from either the left end of the base or the right end, but we get considerable inaccuracies either way. The sum of the areas of the rectangles on the left will be obviously greater than our target, and the sum of the areas of the rectangles on the right will be obviously less. We could take a deep breath, sum the areas of the little rectangles in each case and then average our results. Let's try that and see what we get.

$A_{\text{rectangle}} = bh$, so the area of each little rectangle will equal the product of its base and its height. The length of each base is $\frac{1}{4}$ since we divided the segment $[0, 1]$ into 4 equal segments. The height of each rectangle is determined by the function $y = x^2$, so, taking the height on the right side of each little rectangle, we have

$$A_{\text{under the parabola}} = A_{\text{rectangle1}} + A_{\text{rectangle2}} + A_{\text{rectangle3}} + A_{\text{rectangle4}}$$

$$= \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{2}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{4}{4}\right)^2$$

$$= \frac{1}{4}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2\right]$$

$$= \frac{1}{4}\cdot\frac{1}{4^2}\,[1^2 + 2^2 + 3^2 + 4^2]$$

$$= \frac{1}{64}\,[30]$$

$$= \frac{30}{64} = .46875 \text{, roughly speaking, a little less than } \frac{1}{2}$$

Now let's sum up the little rectangles using the height on the left side:

$A_{\text{under the parabola}} = A_{\text{rectangle1}} + A_{\text{rectangle2}} + A_{\text{rectangle3}} + A_{\text{rectangle4}}$

$$= \frac{1}{4}\cdot\left(\frac{0}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{2}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$

$$= \frac{1}{4}\left[\left(\frac{0}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right]$$

$$= \frac{1}{4}\cdot\frac{1}{4^2}\,[0^2 + 1^2 + 2^2 + 3^2]$$

$$= \frac{1}{64}\,[14]$$

$$= \frac{14}{64} = .21875 \text{, roughly speaking, a little less than } \frac{1}{4}$$

Let's average 'em, and see what we get: $\frac{.46875 + .21875}{2} = \frac{0.6875}{2} = .34375$, a little more than $\frac{1}{3}$

We could maybe get a little more accurate if we divide the segment [0, 1] into 5 equal segments, make rectangles with bases all equal to $\frac{1}{5}$ and use the distance between the x-axis and the parabola on the right side of each rectangle for the height, then add up the areas of the 5 little rectangles and get an approximation of the area under the parabola between 0 and 1. As before, we get

$A_{\text{under the parabola}} = A_{\text{rectangle1}} + A_{\text{rectangle2}} + A_{\text{rectangle3}} + A_{\text{rectangle4}} + A_{\text{rectangle5}}$

$$= \frac{1}{5}\cdot\left(\frac{1}{5}\right)^2 + \frac{1}{5}\cdot\left(\frac{2}{5}\right)^2 + \frac{1}{5}\cdot\left(\frac{3}{5}\right)^2 + \frac{1}{5}\cdot\left(\frac{4}{5}\right)^2 + \frac{1}{5}\cdot\left(\frac{5}{5}\right)^2$$

$$= \frac{1}{5}\left[\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{5}{5}\right)^2\right]$$

$$= \frac{1}{5}\cdot\frac{1}{5^2}\,[1^2 + 2^2 + 3^2 + 4^2 + 5^2]$$

$$= \frac{55}{125} = 0.44$$

And on the low side:

$$= \frac{1}{5} \cdot \left(\frac{0}{5}\right)^2 + \frac{1}{5} \cdot \left(\frac{1}{5}\right)^2 + \frac{1}{5} \cdot \left(\frac{2}{5}\right)^2 + \frac{1}{5} \cdot \left(\frac{3}{5}\right)^2 + \frac{1}{5} \cdot \left(\frac{4}{5}\right)^2$$

$$= \frac{1}{5} \left[\left(\frac{0}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right]$$

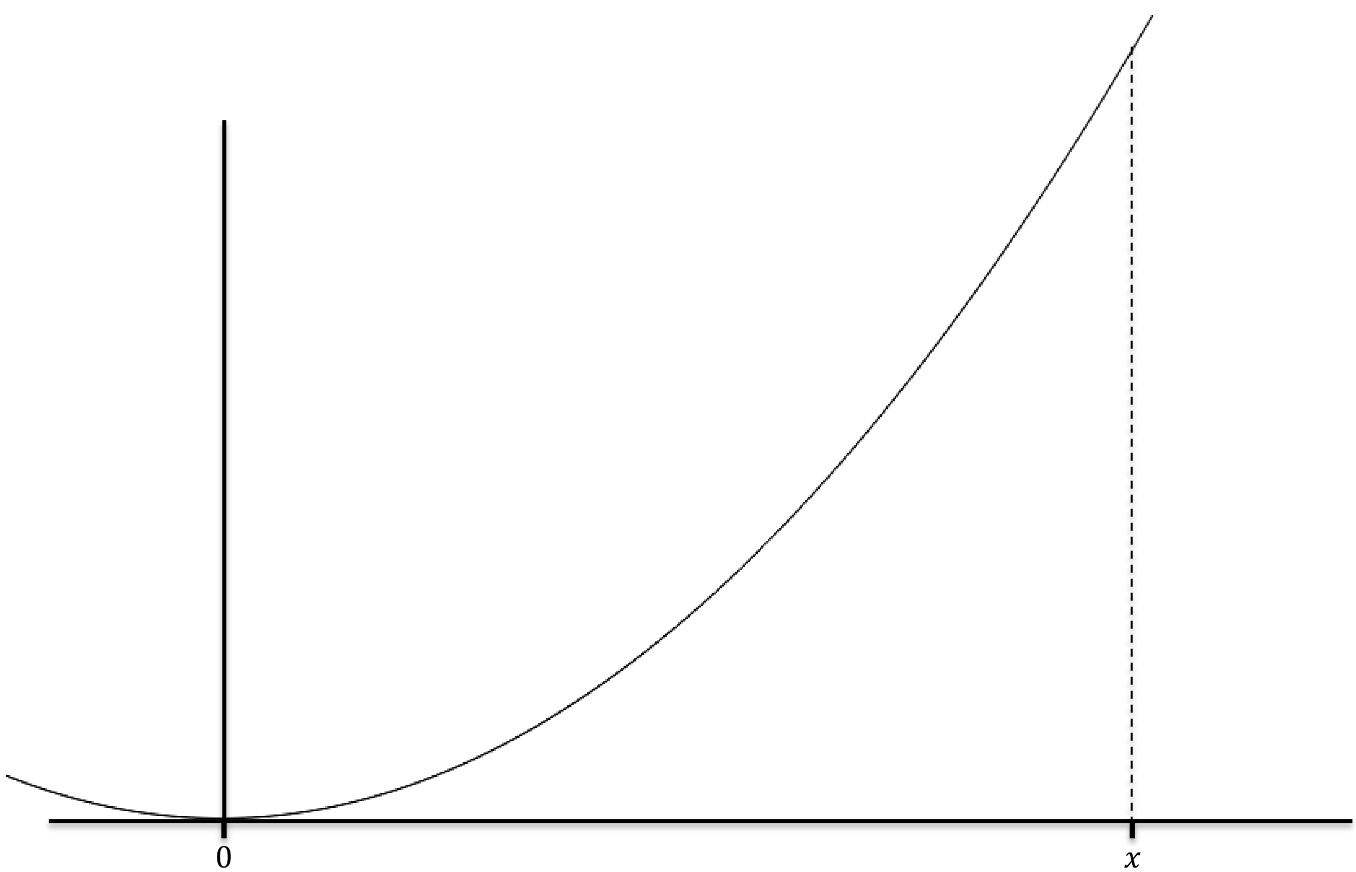
$$= \frac{1}{5} \cdot \frac{1}{5^2} \left[0^2 + 1^2 + 2^2 + 3^2 + 4^2\right]$$

$$= \frac{30}{125} = 0.24$$

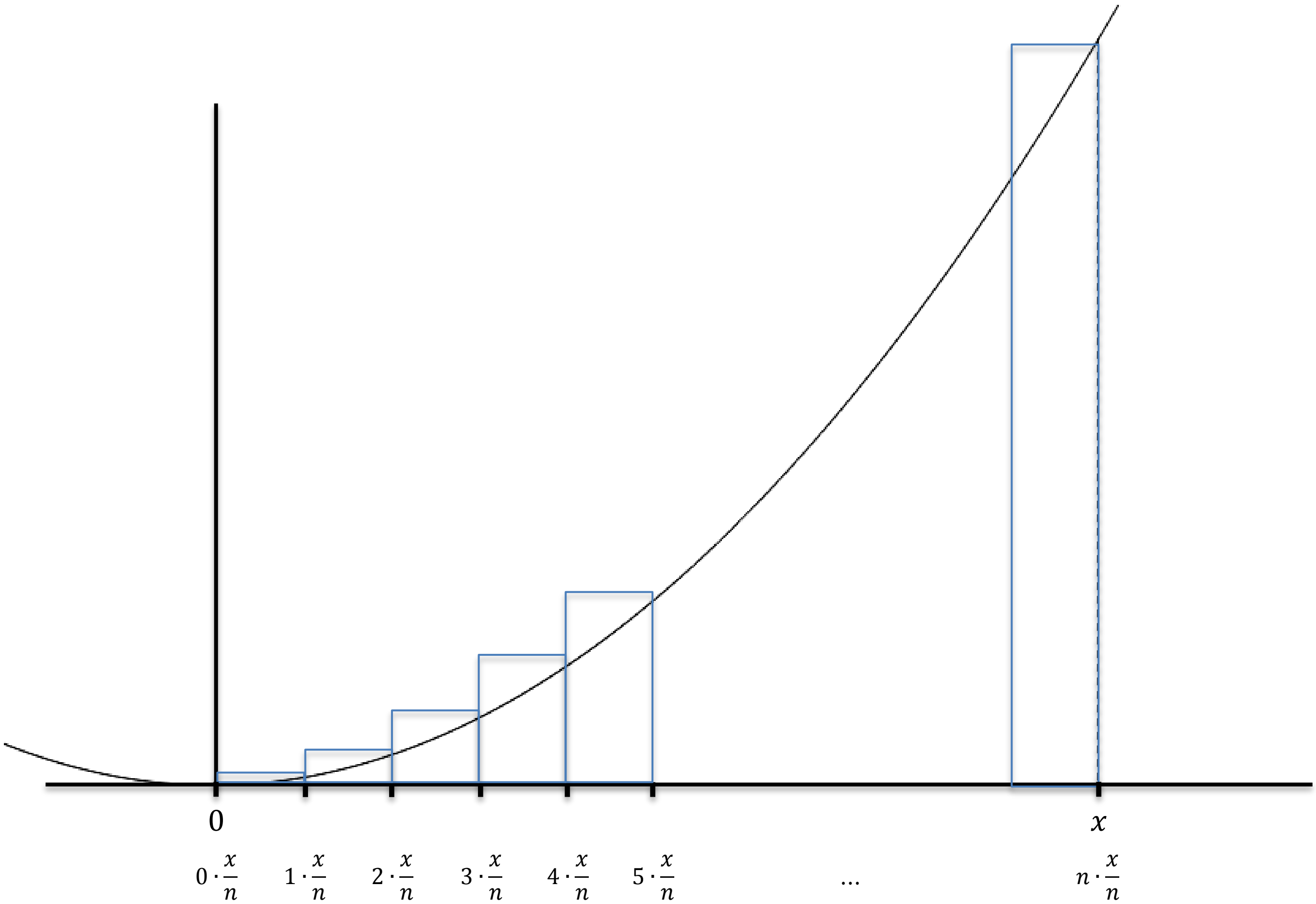
Averaging our two sums, we get .34, a little bit more than $\frac{1}{3}$, and even closer to it.

We should begin to suspect that $\frac{1}{3}$ square unit might be the ultimately accurate answer to our problem. Beyond that, we should be seeing some patterns in our calculations in the workings of the distributive law and the sum of squares. So let's attempt a general description of the problem:

Say, instead of going out on the x-axis to a specific point at 1, we go all the way out to some general point at x.



And say, also, that instead of dividing the segment $[0, x]$ into 4 or 5 equal segments, we divide it into n equal segments, each of which is equal to $\frac{x}{n}$.



Let's use the right endpoint for finding the height of each rectangle. Then our area is the sum of n rectangles, each having base $\frac{x}{n}$ and height $\left(1 \cdot \frac{x}{n}\right)^2, \left(2 \cdot \frac{x}{n}\right)^2, \left(3 \cdot \frac{x}{n}\right)^2$, and so on, respectively, all the way to $\left(n \cdot \frac{x}{n}\right)^2$.

We have, as before,

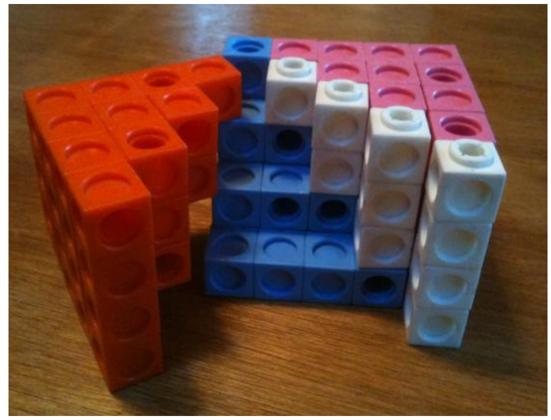
$A_{\text{under the parabola}} = A_{\text{rectangle1}} + A_{\text{rectangle2}} + A_{\text{rectangle3}} + \ldots + A_{\text{rectangle}n}$

$$= \frac{x}{n} \cdot \left(1 \cdot \frac{x}{n}\right)^2 + \frac{x}{n} \cdot \left(2 \cdot \frac{x}{n}\right)^2 + \frac{x}{n} \cdot \left(3 \cdot \frac{x}{n}\right)^2 + \cdots + \frac{x}{n} \cdot \left(n \cdot \frac{x}{n}\right)^2$$

$$= \frac{x}{n} \left[\left(1 \cdot \frac{x}{n}\right)^2 + \left(2 \cdot \frac{x}{n}\right)^2 + \left(3 \cdot \frac{x}{n}\right)^2 + \cdots + \left(n \cdot \frac{x}{n}\right)^2 \right]$$

$$= \frac{x}{n} \cdot \left(\frac{x}{n}\right)^2 \left[1^2 + 2^2 + 3^2 + \cdots + n^2\right]$$ *Distributive Law*

At this point, it comes in very handy to know that the sum of n consecutive squares, $1^2 + 2^2 + 3^2 + \cdots + n^2$, is equal to $\frac{(n)(n+1)(2n+1)}{6}$



See "Sum of a Series of Consecutive Squares" for a detailed step-by-step.

We substitute that into the sum of squares part of our equation above and get

$$= \quad \frac{x}{n} \cdot \left(\frac{x}{n}\right)^2 \left[\frac{(n)(n+1)(2n+1)}{6}\right]$$

$$= \quad \frac{x^3}{n^3} \left[\frac{2n^3+3n^2+n}{6}\right]$$

We take advantage of fraction multiplication and the commutative law for multiplication (switching the 6 and the n^3) so we can deal with all the n terms at once and divide:

$$= \quad \frac{x^3}{6} \left[\frac{2n^3+3n^2+n}{n^3}\right]$$

$$= \quad \frac{x^3}{6} \left[\frac{2n^3}{n^3} + \frac{3n^2}{n^3} + \frac{n}{n^3}\right]$$

$$= \quad \frac{x^3}{6} \left[2 + \frac{3}{n} + \frac{1}{n^2}\right]$$

And suppose further, since we're trying to be very accurate, we make n be a very big number, an infinitely big number. (With infinitely many divisions we should expect to have an infinitely fine approximation.)

Letting n be infinitely large makes things wonderfully simple because as n goes to infinity, both $\frac{3}{n}$ and $\frac{1}{n^2}$ go to zero. We then have

$$= \quad \frac{x^3}{6} [2]$$

$$= \quad \frac{x^3}{3}$$

Now we can be entirely confident that the area under the parabola $y = x^2$ between 0 and 1 is exactly $\frac{1}{3}$ square unit: $\frac{x^3}{3} = \frac{(1)^3}{3} = \frac{1}{3}$

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