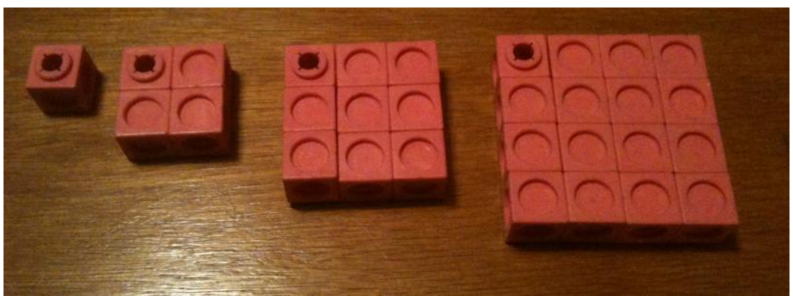
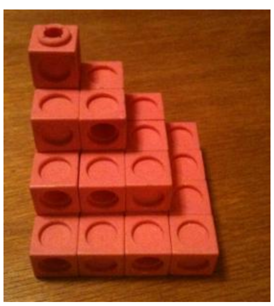
Sum of a Series of Consecutive Squares

We want to end up with a formula for the sum of consecutive squares of whole numbers from 1 to n. The following is an exploration, not really a derivation, but its aim is to make a derivation seem perfectly reasonable.

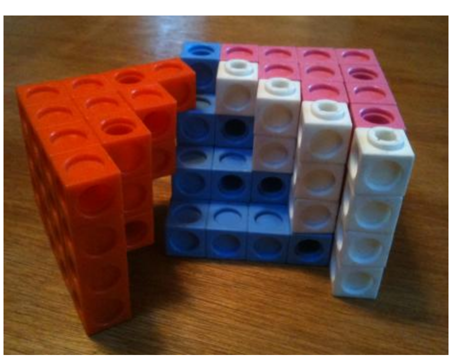
1. Let p = the sum of a series of squares of consecutive whole numbers from 1 to n: $1^2 + 2^2 + 3^2 + \cdots + n^2$
2. Let a = the sum of an arithmetic series of consecutive whole numbers from 1 to n: $1 + 2 + 3 + \cdots + n$
3. When we find p in terms of n, we'll have our formula.
4. We know from our explorations with multilink blocks that we can take a sum of a series of consecutive squares

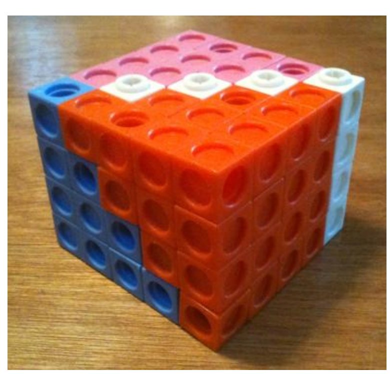


and assemble it into a right pyramid.



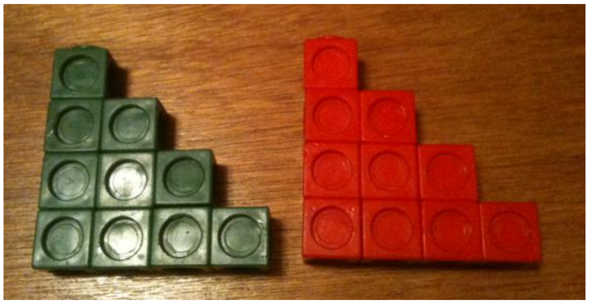
If we make 3 of these right pyramids ($3p$) and fit them together in a certain way, along with an arithmetic series (a) , we can make a rectangular prism whose dimensions are $(n) \times (n+1) \times (n+1)$ with $n = 4$.

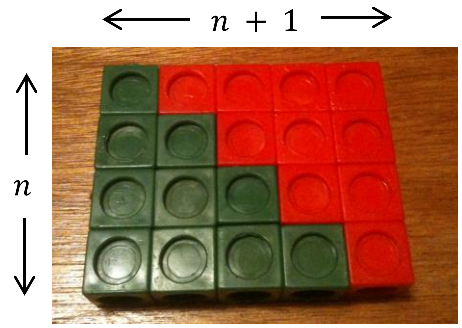




5. We can express this symbolically: $3p + a = n(n+1)(n+1)$

6. To express p in terms of n, we need a value we can substitute for a. From experimenting with pairs of arithmetic series of $1 + 2 + 3 + \cdots + \mathrm{n}$ with multilink blocks (see below for an illustration of $\mathrm{n} = 4$), we notice that if we build an arithmetic series and make a copy of it, it always seems to be true that we can assemble the two chunks into a rectangle with dimensions $(\mathrm{n}) \times (\mathrm{n}+1)$





While we have the multilink blocks out, it would be good to try some other arithmetic series to test our hunch a little more.

7. Remembering that we were letting a = the sum of an arithmetic series of consecutive whole numbers from 1 to n, we can state our hunch in symbolic form: $a = \frac{(n)(n+1)}{2}$

8. Now we can get back to the pyramids and substitute this value for a. We know three sums of squares of $1...n$ plus one sum of an arithmetic series of $1...n$ gives us a rectangular prism with dimensions $n(n+1)(n+1)$. So we have this general equation: $3p + a = n(n+1)(n+1)$ which, when we substitute for a becomes

$$3p + \frac{(n)(n+1)}{2} = n(n+1)(n+1)$$

Subtracting a from both sides, we have

$$3p = (n)(n+1)(n+1) - \frac{(n)(n+1)}{2}$$

$$= \frac{2\,(n)(n+1)(n+1) - (n)(n+1)}{2}$$ *Addition (Subtraction) of Fractions, FFFP*

$$= \frac{(n)(n+1)[2(n+1)-1]}{2}$$ *Commutative Property, Distributive Property (factoring out the $(n)(n+1)$ common to both terms)*

$$= \frac{(n)(n+1)[2n+2-1]}{2}$$

$$= \frac{(n)(n+1)(2n+1)}{2}$$

So $1p = \left(\frac{1}{3}\right)\left[\frac{(n)(n+1)(2n+1)}{2}\right]$

$$\therefore \boxed{p = \frac{(n)(n+1)(2n+1)}{6}}$$

Larry Francis, Southern Oregon ESD
www.soesd.k12.or.us/math
larry_francis@soesd.k12.or.us or 541-951-6830

9/30/2011